Stable Menus of Public Goods

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Harvard University
Motivation

Decision makers often select “public goods” to provide to unit-demand agents:

- Which electives should an online school offer?
- Where should a vending machine company locate identical machines?
- When should I schedule my office hours?
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Common themes in these problems:

- View as matching problem: agents match to favorite available good.
- Each good needs minimum usage to justify existence.
  \[\rightarrow\text{goods’ preferences have complementarities}\]
- No capacity constraints (unlike much of the assignment literature).
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- Each good needs minimum usage to justify existence.
  \[\rightarrow\] goods’ preferences have complementarities
- No capacity constraints (unlike much of the assignment literature).

This talk:

- How to define stability for a matching in this setting?
- Existence of stable outcomes? Strategic considerations?
Related Work

Matching

Public projects

Committee selection
Related Work

Matching
- Existence (Gale + Shapley ’62)
- No complementarities (Hatfield and Kojima ’08)
- Strategyproofness (Dubins + Freedman ’81, Roth ’82)

This work: no capacity constraints, yes complementarities

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- Osheto ’96, Papadimitriou + Schapira + Singer ’08

This work: no money

Committee selection
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Public projects
• Osheto ’96, Papadimitriou + Schapira + Singer ’08
This work: no money

Committee selection
• Aziz et al. ’14, Jiang + Mungala + Wang ’20
This work: no budget
Model

- **n agents**, denoted \( N = \{1, \ldots, n\} \).
- **\( g \) public goods**, denoted \( G = \{1, \ldots, g\} \).
- Each agent \( i \in \{1, \ldots, n\} \) has **complete preferences** \( \succ_i \) over \( G \).
- A **menu** \( M \subseteq G \) induces a **matching**: agent \( i \) uses their favorite good in \( M \).
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A menu \( M \subseteq G \) of public goods is **t-stable** if:
- **t-feasibility**: each provided public good \( \gamma \in M \) is used by \( \geq t \) agents.
- **t-uncontestability**: there do not exist \( t \) “unhappy” agents, and an unprovided public good \( \gamma \in G \setminus M \), such that each of these agents prefers \( \gamma \) over all provided public goods in \( M \).
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- Feasibility \( \rightarrow \) provide fewer public goods
- Uncontestability \( \rightarrow \) provide more public goods
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- **n agents**, denoted $N = \{1, \ldots, n\}$.
- **$g$ public goods**, denoted $G = \{1, \ldots, g\}$.
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A menu $M \subseteq G$ of public goods is **$t$-stable** if:

- **$t$-feasibility**: each provided public good $\gamma \in M$ is used by $\geq t$ agents.
- **$t$-uncontestability**: there do not exist $t$ “unhappy” agents, and an unprovided public good $\gamma \in G \setminus M$, such that each of these agents prefers $\gamma$ over all provided public goods in $M$.

- Feasibility $\rightarrow$ provide fewer public goods
- Uncontestability $\rightarrow$ provide more public goods

- **Menu selection problem** $= (\text{agents, public goods, preferences})$. 
Example

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **t-uncontestability**: $\not\exists t$ "unhappy" agents who prefer $\gamma \in G \setminus M$

\[ \{ \text{t-stable} \} \]
Example

- **$t$-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **$t$-uncontestability**: $\forall$ $t$ "unhappy" agents who prefer $\gamma \in G \setminus M$ \(\{ t\text{-stable} \}\)
Example

- **$t$-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **$t$-uncontestability**: $\nexists$ $t$ “unhappy” agents who prefer $\gamma \in G \setminus M$

### Example

$t = 4$

$n = 9$ agents

$g = 3$ goods

<table>
<thead>
<tr>
<th>Agents</th>
<th>$3 \times 1 \succ 2 \succ 3$</th>
<th>$3 \times 2 \succ 3 \succ 1$</th>
<th>$3 \times 3 \succ 1 \succ 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>${1}$</td>
<td>${1, 2}$</td>
<td>${1, 2, 3}$</td>
</tr>
</tbody>
</table>

$t$-stable
- **$t$-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
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$\emptyset$ not $t$-stable:

$\iff t$-contestable: $9 \geq t$ agents prefer 1 over $\emptyset$

- \{1\}
- \{1, 2\}
- \{1, 2, 3\}
$t$-feasibility: each provided good $\gamma \in M$ used by $\geq t$ agents.

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Example

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **t-uncontestability**: $\not \exists t$ “unhappy” agents who prefer $\gamma \in G \setminus M$ \{t-stable\}

Example

$t = 4$
$n = 9$ agents
$g = 3$ goods

Agents

| 3 \times 1 \succ 2 \succ 3 |
| 3 \times 2 \succ 3 \succ 1 |
| 3 \times 3 \succ 1 \succ 2 |

- $\emptyset$ not t-stable:
  - $\rightarrow$ t-contestable: $9 \geq t$ agents prefer 1 over $\emptyset$
- $\{1\}$ not t-stable:
  - $\rightarrow$ t-contestable: $6 \geq t$ agents prefer 3 over 1
- $\{1, 2\}$ not t-stable:
  - $\rightarrow$ t-infeasible: only 3 < $t$ agents use 2
- $\{1, 2, 3\}$
Example

- **$t$-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **$t$-uncontestability**: $\nexists$ $t$ “unhappy” agents who prefer $\gamma \in G \setminus M$ \(\{t\text{-stable}\}\)

Example

$t = 4$

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Agents

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3 \times 1 & \succ 2 \succ 3 \\
3 \times 2 & \succ 3 \succ 1 \\
3 \times 3 & \succ 1 \succ 2
\end{align*}

<table>
<thead>
<tr>
<th>Bundle</th>
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<td>$\emptyset$</td>
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<td>${1, 2, 3}$</td>
<td>not $t$-stable: $\leftrightarrow t$-infeasible: only $3 &lt; t$ agents use each good</td>
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**Example**

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **t-uncontestability**: $\forall t$ “unhappy” agents who prefer $\gamma \in G \setminus M$ \(\} t\)-stable

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<table>
<thead>
<tr>
<th>Menu</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
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No $t$-stable menu exists!
Example

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
- **u-uncontestability**: $\not\exists u$ “unhappy” agents who prefer $\gamma \in G \setminus M$ \( (t, u) \)-stable

Example

\[
\begin{align*}
\text{t = 4, u = 7} & \\
\text{n = 9 agents} & \\
\text{g = 3 goods} & \\
\text{Agents} & \begin{align*}
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3 \times 1 & \succ 2 \succ 3 \\
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Example

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
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**Example**

- $t = 4$, $u = 7$
- $n = 9$ agents
- $g = 3$ goods

Agents

<table>
<thead>
<tr>
<th>3 x 1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>3 x 2</td>
<td>3</td>
<td>1</td>
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<td>3 x 3</td>
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\[
\emptyset \quad \text{not \((t, u)\)-stable:}
\]

\[
\leftrightarrow \quad \text{u-contestable: } 9 \geq u \text{ agents prefer 1 over } \emptyset
\]

- $\{1\}$
- $\{1, 2\}$
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**Example**

- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
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**(t, u)-stable**

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**Example**

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| Agents | $3 \times 1 \succ 2 \succ 3$
|--------|------------------|
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- **t-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
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- **$t$-feasibility**: each provided good $\gamma \in M$ used by $\geq t$ agents.
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$\{1\}$ ($t$, $u$)-stable.

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When $u/t$ sufficiently large, $(t, u)$-stable menus exist.
Overview

We are interested in questions of:

**Existence**

↔ For which $t, u$ do $(t, u)$-stable menus exist for all menu selection problems?

**Strategyproofness**

↔ When existence guaranteed, for which $g, t, u$ is there a SP mechanism?
We are interested in questions of:

**Existence**

\[ \Leftrightarrow \text{For which } t, u \text{ do } (t, u)\text{-stable menus exist for all menu selection problems?} \]

- When \( u = \infty \) and \( t = 0 \), all menus trivially \((t, u)\text{-stable.}\)

**Strategyproofness**

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- For which \( t, u \) do \((t, u)\)-stable menus exist for all menu selection problems?
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    - Tight characterization for \( g \leq 6 \)

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- \( g = 2 \): simple anonymous SP mechanism
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**Strategyproofness**

→ When existence guaranteed, for which $g, t, u$ is there a SP mechanism?

- $g = 2$: simple anonymous SP mechanism
- $g = 3, 4, 5, 6$: impossibility result (no anonymous SP mechanism)
Existence
Theorem.
Let $g \geq 3$ and $u \leq t - 2$. Then there exists a menu selection problem with no $(t, u)$-stable menu.

Proof sketch.
One can check these agents have no $(t, u)$-stable menu:

- $t - 1 \times 1 \succ 2 \succ 3$
- $t - 1 \times 2 \succ 3 \succ 1$
- $t - 1 \times 3 \succ 1 \succ 2$

Proposition.
Let $g \geq 2$ and $u > g(t - 1)$. Then for all menu selection problems, there exists a $(t, u)$-stable menu.

Proof sketch.
Let $M := \{ \gamma \in G : \exists t \text{ agents with favorite good } \gamma \}$. One can check $M$ is $(t, u)$-stable.
Simple bounds

Lower bound

**Theorem.** Let $g \geq 3$ and $u \leq 2t - 2$. Then there exists a menu selection problem with no $(t, u)$-stable menu.

Upper bound
**Lower bound**

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**Upper bound**
Simple bounds

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**Proof sketch.** Let \( M := \{\gamma \in G : \exists t \text{ agents with favorite good } \gamma\} \). One can check \( M \) is \((t, u)\)-stable.
To guarantee existence of stable menus, simple bounds say:

- Necessary: $u \geq 2t - 1$.
- Sufficient: $u \geq g(t - 1) + 1$.  

\{ gap of factor of $\sim g$ \}

This simple lower bound is tight for $g = 3, 4, 5, 6$:

**Theorem.** Let $g \in \{3, 4, 5, 6\}$ and $u \geq 2t - 1$. Then every menu selection problem has a $(t, u)$-stable menu.

(We'll talk about $g \geq 7$ later...)

**Proof sketch.**

- $g = 3, 4$: analyze greedy algorithm. Analyzing cycle reveals stable menu.
- $g = 5, 6$: solve computationally. Using structural insights, reduce to polyhedra covering problem, $\rightarrow$ 1 week on Harvard cluster using SMT solver.
Tight characterization for $g \leq 6$

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\]

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\]

This simple lower bound is tight for $g = 3, 4, 5, 6$:

**Theorem.** Let $g \in \{3, 4, 5, 6\}$ and $u \geq 2t - 1$. Then every menu selection problem has a $(t, u)$-stable menu.

(We’ll talk about $g \geq 7$ later...)

**Proof sketch.**

- $g = 3, 4$: analyze **greedy algorithm**. Analyzing cycle reveals stable menu.
- $g = 5, 6$: solve **computationally**. Using structural insights, reduce to polyhedra covering problem $\leftrightarrow 1$ week on Harvard cluster using SMT solver.
Encode menu selection problem as

\[ x \in \mathbb{R}^g. \]

Example

Menu selection problem:

- \( 2 \times 1 \succ 2 \succ 3 \)
- \( 3 \times 2 \succ 1 \succ 3 \)

\[ \rightarrow \text{gives vector } x = (2, 0, 0, 3, 0, 0). \]
Sidenote: reduction to polyhedra covering problem

- Encode menu selection problem as

\[ x \in \mathbb{R}^{g^l}. \]

- Construct polyhedron \( P_{M}^{t,u,g} \) s.t.

\[ M \text{ is } (t, u)\text{-stable} \iff x \in P_{M}^{t,u,g}. \]

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  \[ M \text{ is (}t,u\text{-)stable } \iff x \in P_{M}^{t,u,g}. \]

- There exists a stable menu \( M \) for a menu selection problem \( x \) if and only if
  \[ x \in \bigcup_{M \subseteq G} P_{M}^{t,u,g}. \]

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Menu selection problem:
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- All menu selection problems have stable menus if and only if
  \[ \mathbb{Z}^g_{\geq 0} \subseteq \bigcup_{M \subseteq G} P^t,u,g_M. \]

---

Example

Menu selection problem:
- \( 2 \times 1 \succ 2 \succ 3 \)
- \( 3 \times 2 \succ 1 \succ 3 \)

\( \iff \) gives vector \( x = (2, 0, 0, 3, 0, 0). \)
Sidenote: reduction to polyhedra covering problem

- Encode menu selection problem as
  \[ x \in \mathbb{R}^{g!}. \]

- Construct polyhedron \( P_{M}^{t,u,g} \) s.t.
  \( M \) is \((t, u)\)-stable \(\iff\) \( x \in P_{M}^{t,u,g} \).

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- All menu selection problems have stable menus if and only if
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Example

Menu selection problem:
- \( 2 \times 1 \gg 2 \gg 3 \)
- \( 3 \times 2 \gg 1 \gg 3 \)
\(\iff\) gives vector \( x = (2, 0, 0, 3, 0, 0) \).

How to test if \( \{1, 2\} \) is \((t, u)\)-stable?

\( t\)-feasibility:
- \( \langle (1, 1, 0, 0, 1, 0), x \rangle \geq t \)
  types taking 1
- \( \langle (0, 0, 1, 1, 0, 1), x \rangle \geq t \)
  types taking 2

\( u\)-defendability:
- \( \langle (0, 0, 0, 0, 1, 1), x \rangle \leq u \)
  types demanding 3
Encode menu selection problem as

\[ x \in \mathbb{R}^g. \]

Construct polyhedron \( P_{M}^{t,u,g} \) s.t.

\[ M \text{ is } (t, u)-\text{stable} \iff x \in P_{M}^{t,u,g}. \]

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Example

Menu selection problem:

- \( 2 \times 1 \succ 2 \succ 3 \)
- \( 3 \times 2 \succ 1 \succ 3 \)

\[ \iff \text{gives vector } x = (2, 0, 0, 3, 0, 0). \]

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
t \\
t \\
-u + 1 \\
\end{pmatrix}
\]

\[ P_{\{1, 2\}}^{t,u,g} := \{ v : Av \geq b \} \text{ encodes for which menu selection problems } \{1, 2\} \text{ is stable.} \]
**Theorem.** Let $g \geq 7$ and $u \leq 23\lfloor \frac{t-1}{11} \rfloor$. Then there exists a menu selection problem with no $(t, u)$-stable menu. (cf. $u \geq 2t - 1 \iff$ existence when $g \leq 6$)
Beyond $g \geq 7$

**Theorem.** Let $g \geq 7$ and $u \leq 23\lfloor \frac{t-1}{11} \rfloor$. Then there exists a menu selection problem with no $(t, u)$-stable menu. (cf. $u \geq 2t - 1 \Leftrightarrow$ existence when $g \leq 6$)

Set $x := \lfloor \frac{t-1}{11} \rfloor$. Then the following 70x agents have no $(t, u)$-stable menu:

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- Simplified and cleaned from counterexample found by SMT solver.
### Beyond $g \geq 7$

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- Simplified and cleaned from counterexample found by SMT solver.
- Also have somewhat improved upper bound: $u \geq (g-2)(t - 1) + 1$.  


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- Simplified and cleaned from counterexample found by SMT solver.
- Also have somewhat improved upper bound: $u \geq (g-2)(t-1) + 1$.
- When $g \geq 7$, existence question open for $23\left\lfloor \frac{t-1}{11} \right\rfloor < u \leq (g-2)(t-1)$. 
  \[ \sim g \text{ gap} \]
Strategyproofness
Fix $g, t, u$ such that every menu selection problem has a stable menu. Does there exist a strategyproof mechanism

$$\mathcal{M} : (\text{menu selection problem}) \mapsto (t, u)\text{-stable menu?}$$
Fix $g, t, u$ such that every menu selection problem has a stable menu.

Does there exist a strategyproof mechanism

$\mathcal{M} : (\text{menu selection problem}) \mapsto (t, u)$-stable menu?

**Theorem.** For $g = 2$, there exists an anonymous SP mechanism.
Fix $g, t, u$ such that every menu selection problem has a stable menu.

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$$M : (\text{menu selection problem}) \mapsto (t, u)\text{-stable menu?}$$

**Theorem.** For $g = 2$, there exists an anonymous SP mechanism.

**Proof sketch.** Do a majority vote (paying attention when to offer two or zero goods instead).
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**Theorem.** For $g = 3, 4, 5, 6$, there is no anonymous SP mechanism.
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**Proof sketch.** Do a majority vote (paying attention when to offer two or zero goods instead).

**Theorem.** For $g = 3, 4, 5, 6$, there is no anonymous SP mechanism.

**Proof sketch.** Given voting problem, carefully transform into menu selection problem and invoke Gibbard–Statterthwaite (transform so that unanimity implied by stability). Challenge: menu selection problem should only have singletons as stable menus.
Takeaways

• We introduce a new model for a matching market, with complementarities and no capacity constraints.
• For $g \leq 6$, we provide a tight characterization for when stable menus exist.
• For $g \geq 7$, we provide lower and upper bounds for when stable menus exist.
• For $3 \leq g \leq 6$, there are fundamental barriers for strategyproofness.
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Thank you! Questions?